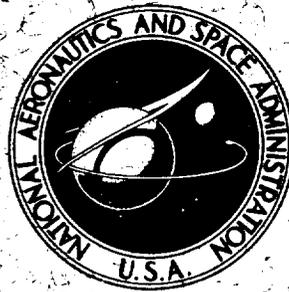


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**PRIMER VECTOR THEORY AND APPLICATIONS**

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# PRIMER VECTOR THEORY AND APPLICATIONS

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## SUMMARY

The results of earlier studies establishing the criteria for an optimal impulsive trajectory and for improving a reference impulsive trajectory are used in this report to develop the necessary conditions for a general, two-body, optimal, N-impulse solution. A general differential cost function is developed which defines the gradient structure for any set of boundary conditions when the applicable constraints are specified.

Example problems are presented to illustrate the generality of the solution; these problems concern fixed-end-condition transfer, orbit-to-orbit transfer, and generalized rendezvous.

Use of a penalty function approach (cost well) enables the establishment of inequality constraints on both state and control variables. The periapsis constraint and a lighting constraint are discussed specifically.

## INTRODUCTION

In the mid to late 1960's, the solution of an optimal trajectory based on replacing the thrust with an impulse received considerable attention as a potentially rapid and reliable method for performing mission design studies. This renewed interest in impulse solutions was prompted by the extensive work of Lawden (ref. 1). Lawden developed the necessary conditions for an optimal impulsive trajectory by examining the limiting conditions on an optimal finite thrust solution wherein the thrust magnitude was unconstrained but bounded between a maximum and a minimum value. The results he obtained are known today as Lawden's necessary conditions for an optimal impulse trajectory. These results specify the conditions that must be satisfied by the primer vector and its derivative on an impulsive trajectory that is considered optimal.

In early 1968, Lion and Handelsman (ref. 2) established the criteria whereby a reference impulsive trajectory could be improved (i.e., whereby a decrease in the cost function or the sum of the magnitudes of the applied impulses could be accomplished). These criteria established the conditions under which an additional intermediate impulse or an initial or final coast would improve the solution. Using Lawden's



necessary conditions, they developed a first-order differential (a gradient vector) of the cost function with respect to the intermediate position vectors and time. With this development, the mechanism for systematically determining the minimum cost for an N-impulse trajectory was created.

In late 1968, Jezewski and Rozendaal perceived the opportunity to generate an efficient method for computing two-body, optimal N-impulse trajectories by combining Lion and Handelsman's gradient vector with a conjugate gradient iterator. A computer program known as the Optimal, Multi, Impulse Rendezvous (OMIR) program was produced. Fixed-time, N-impulse (as many as six impulses), optimal trajectories (ref. 3) could be efficiently and quickly generated with this algorithm.

Boundary conditions (open time, initial and terminal constraints, rendezvous, etc.) were extended with the generation of a general differential cost function. This function defines the gradient structure of the cost function for any set of boundary conditions when the applicable constraints are specified. Inequality constraints (initial, intermediate, and terminal) on both state and control variables were implemented into the OMIR program by a penalty function approach (cost well) (ref. 4). By this means, completely general, two-body, N-impulse, optimal trajectories can be generated for any set of constraints that can be expressed mathematically.

This report consists of material used by the author in a series of lectures on primer vector theory at the NASA Lyndon B. Johnson Space Center in the fall of 1970 and the spring of 1971. These lectures were designed to acquaint the users of the OMIR program with the necessary background material on primer vector theory and its application to trajectory problems. Two approaches are used to present the material: the classical calculus of variations and Pontryagin's maximum principle. The approach used for each topic was determined by the ease and clarity with which the material for that topic could be presented.

## DEFINITION

In a rectangular Cartesian coordinate system, the primer vector  $P$  is defined as a vector that is parallel to the thrust vector  $T$  (fig. 1). The symbol  $P$  does not represent the thrust direction cosine vector  $\ell$ , although the optimum value of this vector  $\ell$  can be defined in terms of the primer vector as  $\ell_{opt} = P/|P|$ . For an optimum transfer problem, the solution requires determining the thrust (magnitude and direction), as a function of time, that minimizes some performance index while satisfying a boundary condition (BC) set. Although the thrust magnitude may not be continuous throughout the solution, the magnitude will usually be bounded (fig. 2).

Other names associated with the primer vector and its derivative are Lagrange multipliers, costate vectors, adjoint variables, and influence coefficients. To better understand the primer vector and its derivative, their origin, and their use in determining optimum control, it will be helpful to examine an optimal trajectory for a mass particle moving in a general gravitational field, in a vacuum, for which the thrust-vector time history is not predetermined. If the results obtained from this solution are examined for a thrust that is impulsive ( $|T| \rightarrow \infty$  or the ratio of the burn

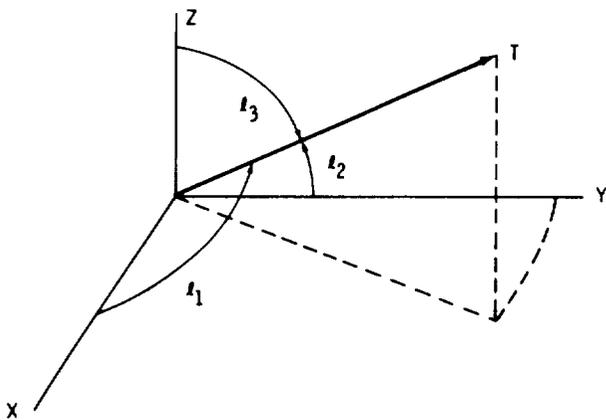


Figure 1.- Thrust-vector orientation.

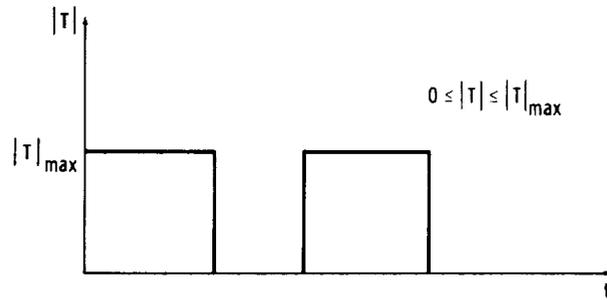


Figure 2.- Bang-bang control.

time to the solution time  $t_b/t_T \rightarrow 0$ ), the necessary conditions on the primer  $P$  and its derivative  $\dot{P}$  will be obtained for an optimal impulsive trajectory.

### LAWDEN'S NECESSARY CONDITIONS

The following material is an interpretation of reference 1, supplemented by references 5 to 7. The state equations for the trajectory problem are

$$\dot{V} = \frac{\beta c \ell}{m} + G \quad (1)$$

$$\dot{R} = V \quad (2)$$

$$\dot{m} = -\beta \quad (3)$$

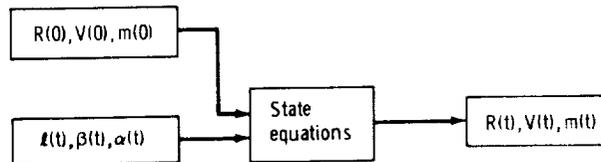
wherein  $R$ ,  $V$ , and  $G$  are the position, velocity, and gravity vectors, respectively. The thrust magnitude is specified by the product  $\beta c$ , where  $\beta$  is the mass flow rate,  $c$  is the exhaust velocity of the engine, and  $m$  is the mass of the vehicle. The direction cosine vector  $\ell$  is constrained by the relationship

$$\ell^T \ell = 1 \quad (4)$$

The mass flow rate is constrained by the relationship

$$\beta (\beta_{\max} - \beta) = \alpha^2 \geq 0 \quad (5)$$

wherein  $\alpha$  is known as a slack variable. The initial conditions  $(R(0), V(0), m(0))$  and the control variables over the solution  $(\ell(t), \beta(t), \alpha(t))$  are needed to obtain a solution of this system of equations  $(R(t), V(t), m(t))$ , as shown in the following diagram.



It is desirable to minimize a performance index  $J$

$$J = J(R_F, V_F, m_F, t_F) \quad (6)$$

subject to the differential constraints (eqs. (1) to (3)) and the algebraic constraints (eqs. (4) and (5)). The subscript  $F$  refers to an evaluation at the final time. Thus, the Hamiltonian function is

$$H = P^T \left( \frac{\beta c \ell}{m} + G \right) + Q^T V - \eta \beta - \mu_1 (\ell^T \ell - 1) - \mu_2 \left[ \beta (\beta_{\max} - \beta) - \alpha^2 \right] \quad (7)$$

wherein the vectors  $P$  and  $Q$  and the scalars  $\eta$ ,  $\mu_1$ , and  $\mu_2$  are time-varying Lagrange multipliers.

The necessary conditions for an optimal trajectory are as follows.

$$\dot{P}^T = -\frac{\partial H}{\partial V} = -Q^T \quad (8)$$

$$\dot{Q}^T = -\frac{\partial H}{\partial R} = -P^T \frac{\partial G}{\partial R} \quad (9)$$

$$\dot{\eta} = -\frac{\partial H}{\partial m} = \frac{\beta c}{m^2} P^T \ell \quad (10)$$

$$0 = -\frac{\partial H}{\partial \ell} = -\frac{\beta c}{m} P^T + 2\mu_1 \ell^T \quad (11)$$

$$0 = -\frac{\partial H}{\partial \beta} = -\frac{c}{m} P^T \ell + \eta + \mu_2 (\beta_{\max} - 2\beta) \quad (12)$$

$$0 = -\frac{\partial H}{\partial \alpha} = -2\mu_2 \alpha \quad (13)$$

From equation (13),  $\mu_2 = 0$  or  $\alpha = 0$  or  $\mu_2 = \alpha = 0$ . If

$$\alpha = 0, \Rightarrow \begin{cases} \beta = 0 \\ \text{or} \\ \beta = \beta_{\max} \end{cases} \quad (14)$$

(See eq. (5).) If  $\alpha \neq 0$  and  $\mu_2 = 0$ ,  $\Rightarrow 0 < \beta < \beta_{\max}$ . Thus, the solution may have the following three types of arcs.

1. Maximum thrust (MT)
2. Null thrust (NT), free-fall arc
3. Intermediate thrust (IT), singular arc



From equation (11), the following conditions are apparent.

1. If  $\beta \neq 0$  and  $\mu_1 \neq 0$ ,  $P$  and  $\ell$  are parallel.
2. If  $\beta = 0$  and  $\mu_1 = 0$ ,  $\ell$  is indeterminate.
3. If  $\mu_1 = 0$  and  $\beta \neq 0$ ,  $P$  vanishes.

Thus,  $P$  and  $\ell$  are parallel; however, are they parallel in the same or in opposing directions? The Weierstrass condition is used to resolve this problem.

The Weierstrass  $E$  function is defined in reference 5 as

$$E = F(X, \dot{X}, t) - F(X^*, \dot{X}^*, t) - \frac{\partial F}{\partial \dot{X}^*}(X^*, \dot{X}^*, t)(\dot{X} - \dot{X}^*) \geq 0 \quad (15)$$

wherein

$$F = \Psi^T (F - \dot{X}) \quad \Psi = \begin{pmatrix} P \\ Q \\ \eta \end{pmatrix} \quad X = \begin{pmatrix} V \\ R \\ m \end{pmatrix} \quad (16)$$

and wherein  $*$  indicates optimal trajectory and  $F$  represents the right-hand side of the state equations. Because the state equations are satisfied on the optimal as well as on the nearby trajectory,  $F(X, \dot{X}, t) = F(X^*, \dot{X}^*, t) = 0$  and equation (15) becomes

$$E = \Psi^T (\dot{X} - \dot{X}^*) \geq 0 \quad (17)$$

Substituting for  $\psi$  and  $X$  and canceling like terms results in

$$\beta \left( \frac{c}{m} P^T \ell - \eta \right) \geq \beta^* \left( \frac{c}{m} P^T \ell^* - \eta \right) \quad (18)$$

Now, examine this equation for the three types of arcs.

1. For the maximum thrust arc ( $\beta = \beta_{\max}$ ), if the optimum mass flow rate  $\beta^* = \beta_{\max}$ , equation (18) reduces to

$$P^T \ell \geq P^T \ell^* \quad (19)$$

This condition will be satisfied for all values of  $\ell^*$  if and only if  $P^T \ell$  takes on its maximum value for all variations of  $\ell$ . Hence,  $P$  must be aligned with  $\ell$  or  $\ell = P/p$  where  $p = |P|$  and

$$\eta \leq \frac{c}{m} |P| \quad (20)$$

2. For the null thrust arc ( $\beta = 0$ ), equation (18) reduces to  $\eta \geq \frac{c}{m} P^T \ell^*$ . For a variable  $\ell^*$ , the maximum value of the right-hand side is obtained when  $P$  and  $\ell^*$  are aligned, or  $(P^T \ell^*)_{\max} = p$ . Thus,

$$\eta \geq \frac{cp}{m} \quad (21)$$

3. For the intermediate thrust arc ( $\beta > \beta^*$  or  $\beta < \beta^*$ ), if  $\ell = \ell^*$ , the inequality in equation (18) is not possible for all permissible values of  $\beta^*$ . However, equality is possible for

$$\eta = \frac{c}{m} P^T \ell \quad (22)$$

With this value for  $\eta$ , a further requirement is  $\eta \geq \frac{c}{m} P^T \ell^*$  or  $\eta \geq \frac{c}{m} p$ . But equation (22) implies that  $\eta \leq \frac{c}{m} p$  and that equality exists only when  $P$  and  $\ell$  are aligned. Thus, for IT arcs,  $P$  and  $\ell$  are aligned and

$$\eta = \frac{c}{m} p \quad (23)$$

The Weierstrass condition is summarized as follows.

1. If  $\beta \neq 0$  and  $p \neq 0$ , thrust is in the direction of the primer.
2. If  $\$$  is defined as

$$\$ = \frac{c}{m}p - \eta \quad (24)$$

then it is necessary that  $\$ > 0$  for an MT arc,  $\$ < 0$  for an NT arc, and  $\$ = 0$  for an IT arc.

Consider now corners separating two types of arcs. The parameters  $P$ ,  $Q$ ,  $\eta$ , and  $H$  are known to be continuous. If  $H$  is to be continuous, then  $\frac{\beta c}{m}p - \eta\beta = \beta\$\$  must be continuous. When  $\beta$  is discontinuous, or the type of arc changes the switch function,  $\$ \equiv 0$ . Taking the time derivative of  $\$$ , one obtains  $\dot{\$} = \frac{c\dot{p}}{m} + \frac{cp\dot{\beta}}{m^2} - \dot{\eta}$ . Substituting from equation (10) for  $\dot{\eta}$ , one obtains

$$\dot{\$} = \frac{c\dot{p}}{m} \quad (25)$$

On an NT arc,  $m$  is constant. Integrating equation (25), one obtains

$$\$ = \frac{cp}{m} + \text{constant} \quad (26)$$

Consider the computation of  $\mu_2$ . From equation (12),  $\mu_2 = \frac{\$}{\beta_{\max} - 2\beta}$  for an MT arc or an NT arc, and  $\mu_2 = 0$  for an IT arc.

Concerning impulsive thrust, if  $\beta \rightarrow \infty$  or  $t_b/t_T \rightarrow 0$ , the maximum thrust is replaced by an impulse of negligible time duration. Under these conditions,  $R$ ,  $G$ , and  $\partial G/\partial R$  are continuous;  $V$  is discontinuous at impulse times. Hence,  $P$ ,  $Q$ , and  $\dot{Q}$  are continuous;  $\dot{Q}$  is discontinuous (involves velocities).

Examine the switch function  $\$$  shown in figures 3(a) and 3(b). At the time of application of an impulse  $t_I$ ,  $\dot{\$} = \dot{\$} = 0$  and  $\$(t_I) = \text{maximum}$ . Another possibility

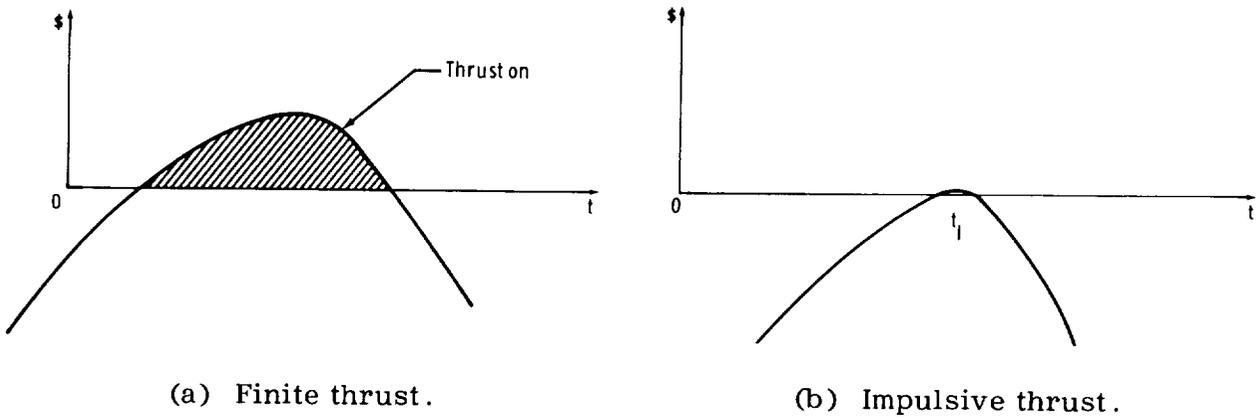


Figure 3.- Switch function.

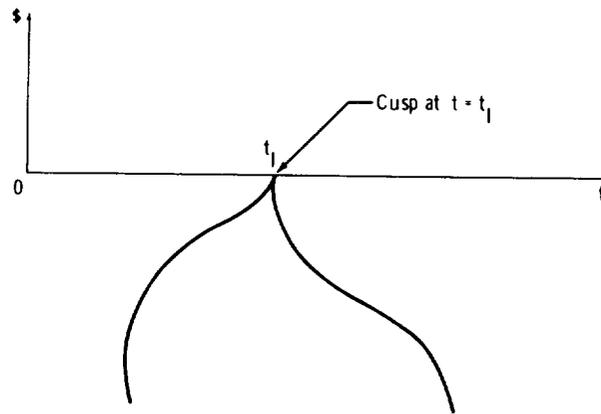


Figure 4.- Switch function, impulsive thrust,  $t = t_I$ .

is shown in figure 4. To prove that  $\dot{\$} = \dot{\$} = 0$  at  $t = t_I$  for an optimal impulse, consider the time derivative of the Hamiltonian function.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = P^T \frac{\partial G}{\partial t} \quad (27)$$

If equation (27) is integrated over an infinitesimal duration of impulse,

$$H(t_I^+) - H(t_I^-) = 0 \quad (28)$$



because the right-hand side of equation (27) is continuous and remains finite. Hence, from equation (28), the Hamiltonian function is continuous across the impulse. But, on an optimal trajectory, from equations (7), (8), and (24)

$$H = P^T G - \dot{P}^T V + \beta \$ \quad (29)$$

But  $\beta \$ = 0$  on an NT arc or on an IT arc; hence,  $\beta \$$  must vanish on both sides of the impulse. Also, because  $P^T G$  is continuous, it follows that

$$\dot{P}^T V(t_I^-) = \dot{P}^T V(t_I^+) \quad (30)$$

or

$$\dot{P}^T (V^+ - V^-) = 0 \quad (31)$$

Because the directions of the primer and the impulse are known to be aligned, or  $V^+ - V^- = \gamma P$  (proportionality factor  $\gamma \neq 0$ ), equation (31) is an indication that

$$P^T \dot{P} = 0 \quad (32)$$

or that  $P \perp \dot{P}$  or  $\dot{p} = 0$ . And, from equation (25),  $\dot{\$}(t_I) = 0$ .

Consider an NT arc joining two impulses, as shown in figure 5. If  $p = a$  at one impulse, then from equation (26),  $\$ = \frac{c}{m}(p - a)$ . Because  $\$ = 0$  at the second

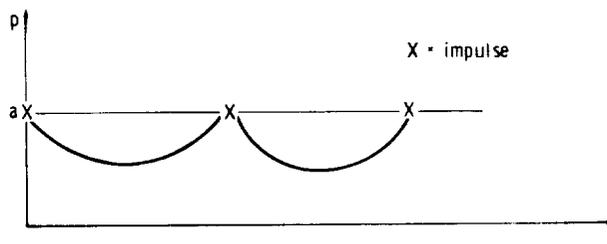


Figure 5.- Primer magnitude.

impulse, it follows that  $p = a$  also at the second impulse. Thus, on an optimal trajectory,  $p = a$  at all impulse points. Between impulses, or on NT arcs,  $p \leq a$ . If  $p > a$ , an impulse would be implied and the trajectory would not be optimal.

To summarize: The following are Lawden's necessary conditions for an optimal impulsive trajectory; each must be satisfied.

1.  $P$  and  $\dot{P}$  are continuous.
2.  $p = a$  at impulse times.
3.  $p < a$  on NT arcs separating impulses.
4.  $\dot{p} = 0$  at all interior impulses.

In these conditions,  $a$  is the magnitude of  $P$  at one impulse.

### CRITERIA FOR AN ADDITIONAL IMPULSE

The criteria for an additional impulse are discussed in the following subsections.

#### Adjoint Equation

The material in this subsection is based on reference 8. Particle motion in a central force field is described by the equations

$$\dot{R} = V \quad (33a)$$

$$\dot{V} = \nabla U(R, t) \quad (33b)$$

wherein  $U$  is the potential and  $\nabla$  is the gradient operator. The variation between a perturbed and a reference trajectory is represented as  $\delta$ ; for example,  $\delta R = R' - R$ . Taking variations of equations (33a) and (33b) results in

$$\left. \begin{aligned} \begin{bmatrix} \delta \dot{R} \\ \delta \dot{V} \end{bmatrix} &= A \begin{bmatrix} \delta R \\ \delta V \end{bmatrix} \\ A &= \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix} \end{aligned} \right\} \quad (34)$$

wherein  $I$  is the identity matrix and  $G$  is the gravity gradient matrix;  $G$  is of the form

$$g_{ij} = \frac{\partial^2 U(R,t)}{\partial R_i \partial R_j} \quad (35)$$

wherein  $i = 1, 2, 3$ , and  $j = 1, 2, 3$ . By definition, the system of equations adjoint to equation (34) is

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\lambda} \end{bmatrix} = -A^T \begin{bmatrix} \varphi \\ \lambda \end{bmatrix} \quad (36)$$

wherein  $\varphi$  and  $\lambda$  are adjoint variables. Writing equations (34) and (36) in second-order form results in

$$\left. \begin{aligned} \ddot{\delta R} &= G \delta R \\ \ddot{\lambda} &= G \lambda \end{aligned} \right\} \quad (37)$$

Premultiplying the first equation by  $\lambda^T$  and the second equation by  $\delta R^T$  and then subtracting, one obtains

$$\lambda^T \delta \ddot{R} - \delta R^T \ddot{\lambda} = \lambda^T G \delta R - \delta R^T G \lambda = 0 \quad (38)$$

If  $\dot{\lambda}^T \delta V$  is added and subtracted, then

$$\lambda^T \delta \dot{V} + \dot{\lambda}^T \delta V - \delta R^T \ddot{\lambda} - \dot{\lambda}^T \delta V = 0 \quad (39)$$

which may be expressed as the total time derivative

$$\frac{d}{dt} (\lambda^T \delta V - \dot{\lambda}^T \delta R) = 0 \quad (40)$$

Integrating this equation, one obtains

$$\lambda^T \delta V - \dot{\lambda}^T \delta R = \text{constant} \quad (41)$$

Equation (41) is known as the adjoint equation. The relationship applies for the interval between any two impulses and will be used extensively in the analysis.

### State Transition Matrix

The state variable perturbations at any two times between impulses are related by the state transition matrix.

$$\begin{bmatrix} \delta R(t) \\ \delta V(t) \end{bmatrix} = \Phi(t, \tau) \begin{bmatrix} \delta R(\tau) \\ \delta V(\tau) \end{bmatrix} \quad (42)$$

wherein  $\Phi(t, \tau)$  is the six-by-six matrix of partial derivatives.

$$\Phi(t, \tau) = \begin{bmatrix} \varphi_{11}(t, \tau) & \varphi_{12}(t, \tau) \\ \varphi_{21}(t, \tau) & \varphi_{22}(t, \tau) \end{bmatrix} \quad (43)$$

For example,  $\varphi_{11}(t, \tau) = \partial R(t) / \partial R(\tau)$ . Note from equation (37) that the adjoint variables are transformed in the same manner as perturbations in the position vector; therefore, the following equation can be written.

$$\begin{bmatrix} \lambda(t) \\ \dot{\lambda}(t) \end{bmatrix} = \Phi(t, \tau) \begin{bmatrix} \lambda(\tau) \\ \dot{\lambda}(\tau) \end{bmatrix} \quad (44)$$

At the impulse times  $t = t_1$  and  $t = t_2$ , the maximum value of the primer magnitude ( $P = \lambda$ ) is chosen to be unity; that is,

$$\left. \begin{aligned} \lambda(t_1) &= \lambda_1 = \Delta V_1 / |\Delta V_1| \\ \lambda(t_2) &= \lambda_2 = \Delta V_2 / |\Delta V_2| \end{aligned} \right\} \quad (45)$$

The BC equations (eq. (45)) together with equation (44), when  $t = t_2$  and  $\tau = t_1$ , uniquely define  $\dot{\lambda}(t_1)$  and  $\dot{\lambda}(t_2)$  if the matrix  $\varphi_{12}$  is nonsingular. Therefore,

$$\dot{\lambda}(t_1) = \varphi_{12}^{-1}(t_2, t_1) \left[ \dot{\lambda}(t_2) - \varphi_{11}(t_2, t_1) \dot{\lambda}(t_1) \right] \quad (46)$$

## COST FUNCTION

The impulsive cost function and its gradient vector under specific and general conditions are discussed in the following subsections.

### Fixed End Conditions and Fixed Transfer Time

Define the cost on an impulsive trajectory  $J$  in the following manner (refs. 2 and 3).

$$J = \sum_{i=1}^N |\Delta V_i| \quad (47)$$

Consider the following reference and perturbed trajectories from an orbit A to an orbit B (fig. 6). At a time  $t_m$  ( $t_1 < t_m < t_2$ ), the position vector  $R_m$  on the reference trajectory is perturbed by an amount  $\delta R_m$ . Two Lambert solutions are obtained between the following position vectors and times on the perturbed trajectory.

1.  $[R(t_1), t_1] \rightarrow [R(t_m) + \delta R_m, t_m]$ .
2.  $[R(t_m) + \delta R_m, t_m] \rightarrow [R(t_2), t_2]$ .

The velocities on the perturbed trajectory differ from those on the reference trajectory as

1.  $\delta V_1$  at time  $t_1$
2.  $\delta V_m^-$  at time  $t_m^-$
3.  $\delta V_m^+$  at time  $t_m^+$
4.  $\delta V_2$  at time  $t_2$

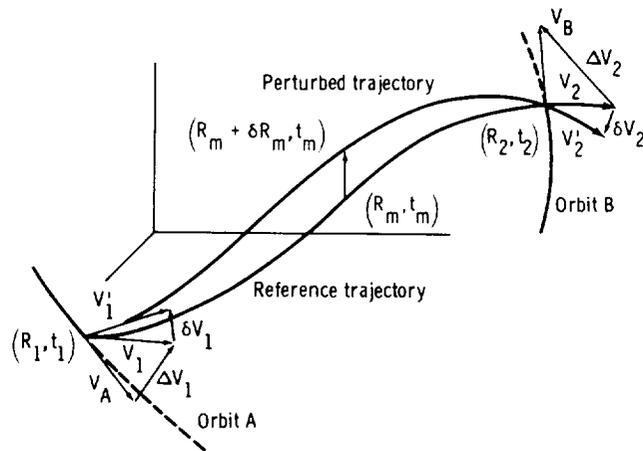


Figure 6.- Reference and perturbed trajectories (fixed conditions), where fixed transfer time  $T = t_2 - t_1$ .

The cost increment to first order between the perturbed  $J'$  trajectory and the reference  $J$  trajectory is defined as

$$dJ = J' - J \quad (48)$$

wherein

$$J = |\Delta V_1| + |\Delta V_2| \quad (49)$$

and

$$J' = |\Delta V_1 + \delta V_1| + |\delta V_m^+ - \delta V_m^-| + |\Delta V_2 - \delta V_2| \quad (50)$$

Digress temporarily and consider the following problem: To first order, what is the difference between the absolute values of a perturbed vector and a reference vector? Let  $df$  equal this difference in the following equation.

$$\begin{aligned}
df &= |X + \delta X| - |X| \\
&= \left[ (X + \delta X)^T (X + \delta X) \right]^{\frac{1}{2}} - \left[ X^T X \right]^{\frac{1}{2}} \\
&= \left[ X^T X \right]^{\frac{1}{2}} \left\{ \left[ \frac{(X + \delta X)^T (X + \delta X)}{X^T X} \right]^{\frac{1}{2}} - 1 \right\} \\
&= \left[ X^T X \right]^{\frac{1}{2}} \left\{ \left[ \left( \ell_X + \frac{\delta X}{|X|} \right)^T \left( \ell_X + \frac{\delta X}{|X|} \right) \right]^{\frac{1}{2}} - 1 \right\} \\
&= |X| \left\{ \left[ 1 + 2 \frac{\delta X^T}{|X|} \ell_X + \dots \right]^{\frac{1}{2}} - 1 \right\} \\
&= |X| \left\{ \left[ 1 + \frac{\delta X^T}{|X|} \ell_X + \dots \right] - 1 \right\} \\
&= \delta X^T \ell_X \tag{51}
\end{aligned}$$

wherein  $\ell_X = X/|X|$ . Hence, if  $\delta R_m$  is small,

$$dJ = \frac{\Delta V_1^T \delta V_1}{|\Delta V_1|} + \left| \delta V_m^+ - \delta V_m^- \right| - \frac{\Delta V_2^T \delta V_2}{|\Delta V_2|} \tag{52}$$

If equation (45) is used,

$$dJ = \lambda_1^T \delta V_1 + \left| \delta V_m^+ - \delta V_m^- \right| - \lambda_2^T \delta V_2 \tag{53}$$

Evaluating the adjoint equation (eq. (41)) at the boundaries of the two Lambert solutions of the perturbed trajectory, one obtains

$$\left. \begin{aligned} \lambda_1^T \delta V_1 - \dot{\lambda}_1^T \delta R_1 &= \lambda_m^{-T} \delta V_m^- - \dot{\lambda}_m^{-T} \delta R_m^- \\ \lambda_m^{+T} \delta V_m^+ - \dot{\lambda}_m^{+T} \delta R_m^+ &= \lambda_2^T \delta V_2 - \dot{\lambda}_2^T \delta R_2 \end{aligned} \right\} \quad (54)$$

If  $R_1$  and  $R_2$  are considered fixed, then

$$\delta R_1 = \delta R_2 = 0 \quad (55)$$

Also,  $\lambda$  and  $\dot{\lambda}$  are evaluated on the reference trajectory and hence are continuous.

$$\left. \begin{aligned} \lambda_m^+ &= \lambda_m^- = \lambda_m \\ \dot{\lambda}_m^+ &= \dot{\lambda}_m^- = \dot{\lambda}_m \end{aligned} \right\} \quad (56)$$

Using equations (55) and (56) in equation (54),

$$\left. \begin{aligned} \lambda_1^T \delta V_1 &= \lambda_m^T \delta V_m^- - \dot{\lambda}_m^T \delta R_m^- \\ \lambda_m^T \delta V_m^+ - \dot{\lambda}_m^T \delta R_m^+ &= \lambda_2^T \delta V_2 \end{aligned} \right\} \quad (57)$$

Solving the first equation for  $\lambda_1^T \delta V_1$  and the second equation for  $\lambda_2^T \delta V_2$ , and using the results in equation (53), one obtains

$$dJ = \dot{\lambda}_m^T \delta R_m^+ - \dot{\lambda}_m^T \delta R_m^- + \left| \delta V_m^+ - \delta V_m^- \right| - \lambda_m^T \left( \delta V_m^+ - \delta V_m^- \right) \quad (58)$$



Now

$$\left. \begin{aligned} \delta R_m^+ &= dR_m^+ - \dot{R}_m^+ dt_m \\ \delta R_m^- &= dR_m^- - \dot{R}_m^- dt_m \end{aligned} \right\} \quad (59)$$

and  $dt_m$  has been chosen to be zero. Because the position vector is continuous,  $\delta R_m^+ = \delta R_m^-$  and equation (58) reduces to

$$dJ = \left| \delta V_m^+ - \delta V_m^- \right| - \lambda_m^T \left( \delta V_m^+ - \delta V_m^- \right) \quad (60)$$

This equation is homogeneous in  $\delta V_m^+ - \delta V_m^-$ . If  $c$  and the unit vector  $\Delta$  are defined as

$$\left. \begin{aligned} c &= \left| \delta V_m^+ - \delta V_m^- \right| \\ \Delta &= \frac{\delta V_m^+ - \delta V_m^-}{c} \end{aligned} \right\} \quad (61)$$

equation (60) reduces to

$$dJ = c \left( 1 - \lambda_m^T \Delta \right) \quad (62)$$

The criteria for an additional impulse can now be established, for if  $|\lambda_m| > 1$ , then  $dJ < 0$  and  $J > J'$ ; or the reference-trajectory cost is greater than the perturbed-trajectory cost. The reference-trajectory cost may be improved by applying an impulse in the direction of  $\lambda_m$  at the time  $t_m$ . To first order, the greatest decrease in cost will be obtained if the intermediate impulse is applied when  $|\lambda_m| = |\lambda_m|_{\max}$  as in figure 7.

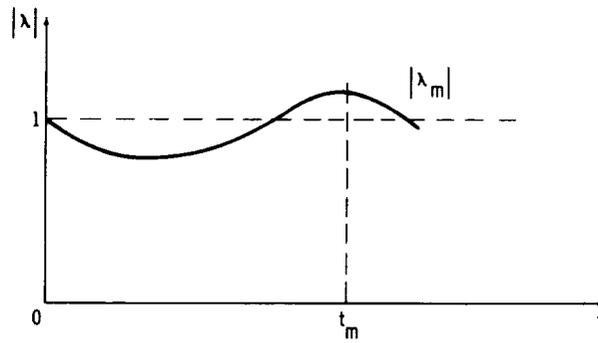


Figure 7.- Primer magnitude characteristics.

If a reference trajectory exists for which  $|\lambda_m| > 1$  at some time  $t_m$ , how much should this trajectory be perturbed and in which direction so as to obtain a perturbed trajectory with a lower cost? From the transition matrix,

$$\left. \begin{aligned} \begin{bmatrix} \delta R_m^- \\ \delta V_m^- \end{bmatrix} &= \Phi(t_m, t_1) \begin{bmatrix} \delta R_1 \\ \delta V_1 \end{bmatrix} \\ \begin{bmatrix} \delta R_m^+ \\ \delta V_m^+ \end{bmatrix} &= \Phi(t_m, t_2) \begin{bmatrix} \delta R_2 \\ \delta V_2 \end{bmatrix} \end{aligned} \right\} \quad (63)$$

If equation (55) is used in equation (63), the following relationships are established.

1.  $\delta R_m^- = \varphi_{12}(t_m, t_1) \delta V_1$ .
2.  $\delta V_m^- = \varphi_{22}(t_m, t_1) \delta V_1$ .
3.  $\delta R_m^+ = \varphi_{12}(t_m, t_2) \delta V_2$ .
4.  $\delta V_m^+ = \varphi_{22}(t_m, t_2) \delta V_2$ .

Eliminating  $\delta V_1$  and  $\delta V_2$  from this set of equations and solving for  $\delta V_m^-$  and  $\delta V_m^+$ , one obtains

$$\delta V_m^- = \varphi_{22}(t_m, t_1) \varphi_{12}^{-1}(t_m, t_1) \delta R_m^- \quad (64)$$

and

$$\delta V_m^+ = \varphi_{22}(t_m, t_2) \varphi_{12}^{-1}(t_m, t_2) \delta R_m^+ \quad (65)$$

By differencing these two equations and recalling that  $\delta R_m^+ = \delta R_m^- = \delta R_m$ , one obtains the following equation.

$$\left( \delta V_m^+ - \delta V_m^- \right) = c \frac{\lambda_m}{|\lambda_m|} = A \delta R_m \quad (66)$$

wherein

$$A = \varphi_{22}(t_m, t_2) \varphi_{12}^{-1}(t_m, t_2) - \varphi_{22}(t_m, t_1) \varphi_{12}^{-1}(t_m, t_1) \quad (67)$$

The result of solving for  $\delta R_m$  from equation (66) is

$$\delta R_m = cA^{-1} \frac{\lambda_m}{|\lambda_m|} \quad (68)$$

As long as  $c$  (the magnitude of the intermediate impulse) is sufficiently small to ensure that the first-order theory holds,  $J' < J$ .

Digress temporarily to consider the following problem: What is a better way of determining the parameter  $c$ ? Consider the differential cost function equation (eq. (48)).

$$dJ = \left| \Delta V_1 + \delta V_1 \right| - \left| \Delta V_1 \right| + \left| \Delta V_2 - \delta V_2 \right| - \left| \Delta V_2 \right| + c \quad (69)$$

The result of retaining second-order terms in the expansion of this function is

$$dJ = \frac{1}{2|\Delta V_1|} \left[ \delta V_1^T \delta V_1 + 2 \delta V_1^T \Delta V_1 - \left( \frac{\delta V_1^T \Delta V_1}{|\Delta V_1|} \right)^2 \right] + \frac{1}{2|\Delta V_2|} \left[ \delta V_2^T \delta V_2 - 2 \delta V_2^T \Delta V_2 - \left( \frac{\delta V_2^T \Delta V_2}{|\Delta V_2|} \right)^2 \right] + c \quad (70)$$

From equations (63) and (68),

$$\left. \begin{aligned} \delta V_1 &= \varphi_{12}^{-1}(t_m, t_1) \delta R_m = c\alpha \\ \delta V_2 &= \varphi_{12}^{-1}(t_m, t_2) \delta R_m = c\beta \end{aligned} \right\} \quad (71)$$

wherein

$$\left. \begin{aligned} \alpha &= \varphi_{12}^{-1}(t_m, t_1) A^{-1} \lambda_m / |\lambda_m| \\ \beta &= \varphi_{12}^{-1}(t_m, t_2) A^{-1} \lambda_m / |\lambda_m| \end{aligned} \right\} \quad (72)$$



By using equation (71) in equation (70), a quadratic in  $c$  can be obtained of the form  $dJ = a_0 + a_1c + a_2c^2$ . Taking the partial derivative of  $dJ$  with respect to  $c$  and solving for  $c$  from  $\partial(dJ)/\partial c = 0$  gives

$$c = -a_1/2a_2 \quad (73)$$

This initial three-impulse trajectory is not necessarily optimum (fig. 8). Although the primer is continuous (eq. (66)) and is a unit vector at each impulse, a cusp may occur in the primer curve because the primer derivative has been neglected. Consider a differential change in the time  $t_m$  and in the position vector  $R_m$ . The two Lambert solutions consist of trajectories between the following position vectors and times.

1.  $[R_1, t_1] \rightarrow [R_m + \delta R_m, t_m + dt_m]$ .
2.  $[R_m + \delta R_m, t_m + dt_m] \rightarrow [R_2, t_2]$ .

wherein  $R_m$  and  $t_m$  now refer to the initial three-impulse trajectory. The differential cost on this initial three-impulse trajectory is

$$dJ = \lambda_1^T \delta V_1 + \left| \delta V_m^+ - \delta V_m^- \right| - \lambda_2^T \delta V_2 \quad (53)$$

The adjoint equations evaluated on the two Lambert solutions (considering that  $\delta R_1 = \delta R_2 = 0$ , that  $\lambda_m$  is continuous, and that  $\lambda$  is not continuous) become

$$\left. \begin{aligned} \lambda_1^T \delta V_1 &= \lambda_m^T \delta V_m^- - \dot{\lambda}_m^{-T} \delta R_m^- \\ \lambda_m^T \delta V_m^+ - \dot{\lambda}_m^{+T} \delta R_m^+ &= \lambda_2^T \delta V_2 \end{aligned} \right\} \quad (74)$$

Using equation (74) in equation (53) results in

$$dJ = \lambda_m^T \delta V_m^- - \dot{\lambda}_m^{-T} \delta R_m^- + \left| \delta V_m^+ - \delta V_m^- \right| - \lambda_m^T \delta V_m^+ + \dot{\lambda}_m^{+T} \delta R_m^+ \quad (75)$$

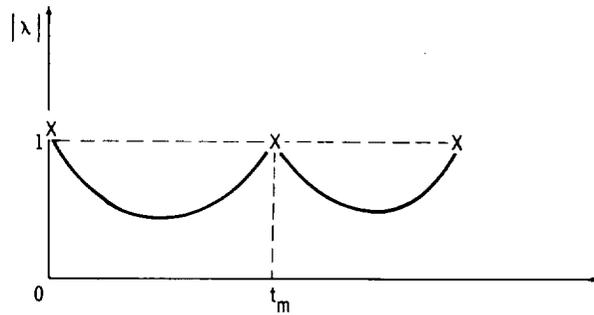


Figure 8.- Primer magnitude (non-optimal) characteristics.

Because  $\lambda_m$  is a unit vector along the interior impulse,

$$+\lambda_m^T (\delta V_m^+ - \delta V_m^-) = |\delta V_m^+ - \delta V_m^-| \quad (76)$$

and the appropriate terms in equation (75) cancel. Removal of these canceled terms leaves

$$dJ = \dot{\lambda}_m^{+T} \delta R_m^+ - \dot{\lambda}_m^{-T} \delta R_m^- \quad (77)$$

But, to first order, the perturbations  $\delta R_m^+$  and  $\delta R_m^-$  are

$$\left. \begin{aligned} \delta R_m^+ &= dR_m - V_m^+ dt_m \\ \delta R_m^- &= dR_m - V_m^- dt_m \end{aligned} \right\} \quad (78)$$

Using equation (78) in equation (77) results in

$$dJ = (\dot{\lambda}_m^+ - \dot{\lambda}_m^-)^T dR_m - (\dot{\lambda}_m^{+T} V_m^+ - \dot{\lambda}_m^{-T} V_m^-) dt_m \quad (79)$$



Because the acceleration is continuous (only a function of position; i.e.,  $\dot{\mathbf{V}} = -\mu\mathbf{R}/|\mathbf{R}|^3$ , where  $\mu$  is the gravitational parameter) on an impulsive trajectory,  $\lambda_m^T \dot{\mathbf{V}}_m$  may be added to and subtracted from equation (79) without its value being changed. The Hamiltonian function  $H$  on a coasting trajectory is  $H = \mathbf{P}^T \mathbf{G} + \mathbf{Q}^T \mathbf{V}$ , which may be written as

$$H = \lambda^T \dot{\mathbf{V}} - \dot{\lambda}^T \mathbf{V} \quad (80)$$

By evaluating equation (80) at  $t_m^+$  and  $t_m^-$  and using the results in equation (79), the cost function becomes

$$dJ = \left( \dot{\lambda}_m^+ - \dot{\lambda}_m^- \right)^T d\mathbf{R}_m + \left( H_m^+ - H_m^- \right) dt_m \quad (81)$$

Thus, equation (81) illustrates that, on an optimal trajectory ( $dJ = 0$ ), the primer vector derivative  $\dot{\lambda}$  and the Hamiltonian function must be continuous across the intermediate impulse.

### General Case

Consider the derivation of a general differential cost function; apply the reference and perturbed trajectories shown in figure 9. The velocities on the perturbed trajectory differ from those on the reference trajectory by the following amounts.

1.  $\delta V_1^-$  at the time  $t_1^-$
2.  $\delta V_1^+$  at the time  $t_1^+$
3.  $\delta V_m^-$  at the time  $t_m^-$
4.  $\delta V_m^+$  at the time  $t_m^+$
5.  $\delta V_2^-$  at the time  $t_2^-$
6.  $\delta V_2^+$  at the time  $t_2^+$

The velocities  $\delta V_1^-$  and  $\delta V_2^+$  are perturbations in the initial and final orbit, respectively, that result from the velocity variation  $\mathbf{V}_A' - \mathbf{V}_A$  in the initial orbit and the velocity variation  $\mathbf{V}_B - \mathbf{V}_B'$  in the final orbit. These variations are caused by consideration of perturbations in the initial and final position vectors that in the previous analysis were assumed to equal zero (eq. (55)).



The first two terms deal with perturbations in the initial orbit; the last two terms, with perturbations in the final orbit. The primer, because it is computed from the reference trajectory, is continuous across the intermediate impulse. Thus,

$\lambda_m^- = \lambda_m^+ = \lambda_m$ . Because  $\lambda_m$  is defined as

$$\lambda_m = \frac{(\delta V_m^+ - \delta V_m^-)}{|\delta V_m^+ - \delta V_m^-|} \quad (85)$$

the third and sixth terms in equation (84) cancel. Also, because the position vector at  $t = t_m$  is required to be continuous, the following first-order equations may be obtained.

$$\left. \begin{aligned} \delta R_m^+ &= dR_m - V_m^+ dt_m \\ \delta R_m^- &= dR_m - V_m^- dt_m \end{aligned} \right\} \quad (78)$$

Therefore, using equations (85) and (78) in equation (84) results in

$$\begin{aligned} dJ &= \dot{\lambda}_1^T \delta R_1^+ - \lambda_1^T \delta V_1^- + (\dot{\lambda}_m^+ - \dot{\lambda}_m^-)^T dR_m \\ &\quad - (\dot{\lambda}_m^{+T} V_m^+ - \dot{\lambda}_m^{-T} V_m^-) dt_m - \dot{\lambda}_2^T \delta R_2^- + \lambda_2^T \delta V_2^+ \end{aligned} \quad (86)$$

Evaluating the Hamiltonian function (eq. (80)) at  $t_m^-$  and  $t_m^+$  results in

$$\left. \begin{aligned} H_m^- &= \lambda_m^{-T} \dot{V}_m^- - \dot{\lambda}_m^{-T} V_m^- \\ H_m^+ &= \lambda_m^{+T} \dot{V}_m^+ - \dot{\lambda}_m^{+T} V_m^+ \end{aligned} \right\} \quad (87)$$

By recalling that  $\dot{V}_m^- = \dot{V}_m^+ = \dot{V}_m$ , equation (87) may be solved for the terms  $\dot{\lambda}^{+T} V_m^+$  and  $\dot{\lambda}^{-T} V_m^-$ . The resulting expressions may be used in equation (86) to obtain

$$\begin{aligned} dJ = & \dot{\lambda}_1^T \delta R_1 - \lambda_1^T \delta V_1^- + \left( \dot{\lambda}_m^+ - \dot{\lambda}_m^- \right)^T dR_m \\ & + \left( H_m^+ - H_m^- \right) dt_m - \dot{\lambda}_2^T \delta R_2 - \lambda_2^T \delta V_2^+ \end{aligned} \quad (88)$$

The perturbation in the initial and final position and velocity vectors can be written to first order as

$$\left. \begin{aligned} \delta R_1^+ &= dR_1 - V_1^+ dt_1 & \delta R_2^- &= dR_2 - V_2^- dt_2 \\ \delta V_1^- &= dV_1 - \dot{V}_1^- dt_1 & \delta V_2^+ &= dV_2 - \dot{V}_2^+ dt_2 \end{aligned} \right\} \quad (89)$$

By combining equations (89) and (88) and by using the Hamiltonian function evaluated at  $t = t_1$  and  $t = t_2$ , one obtains the general differential cost function.

$$\begin{aligned} dJ = & H_1 dt_1 - \left( \lambda_1^T dV_1 - \dot{\lambda}_1^T dR_1 \right) + \left( \dot{\lambda}_m^+ - \dot{\lambda}_m^- \right)^T dR_m \\ & - H_2 dt_2 + \left( \lambda_2^T dV_2 - \dot{\lambda}_2^T dR_2 \right) + \left( H_m^+ - H_m^- \right) dt_m \end{aligned} \quad (90)$$

Equation (90) can be expressed as

$$dJ = \nabla J^T dZ \quad (91)$$

wherein  $\nabla$  is the gradient operator and  $Z$  is a vector of possible control parameters.

$$Z^T = \left[ t_1, R_1^T, V_1^T, t_m, R_m^T, t_2, R_2^T, V_2^T \right] \quad (92)$$

Equation (90) or (91) defines the general differential cost function. From this expression, it can be determined how any two-impulse solution or any two-impulse segment of an N-impulse trajectory can be improved by the following changes.

1. Perturbing the initial variables  $(t_1, R_1, V_1)$
2. Perturbing the final variables  $(t_2, R_2, V_2)$
3. Perturbing the intermediate variables  $(t_m, R_m)$

If a particular variable (i.e., the  $Z_i$ th) is not allowed to be perturbed and hence is fixed, then  $dZ_i = 0$  and the general differential cost function is unaffected.

## EXAMPLE PROBLEMS

### Fixed-End-Condition Transfer

Consider a problem designed to find two-impulse solutions between a fixed initial state and a fixed final state for a transfer time that is open or free. Because intermediate impulses are excluded,  $R_m$  and  $t_m$  are fixed; therefore,

$$dR_m = dt_m = 0 \quad (93)$$

In addition, because the initial position and velocity vector  $(R_1, V_1)$  and the final position and velocity vector  $(R_2, V_2)$  are fixed

$$dR_1 = dR_2 = dV_1 = dV_2 = 0 \quad (94)$$

Using equations (93) and (94) in equation (90) results in a differential cost function of

$$dJ = H_1 dt_1 - H_2 dt_2 \quad (95)$$

Because this problem concerns two-impulse solutions, the Hamiltonian function is constant across the trajectory.

$$H_1 = H_2 = H \quad (96)$$

Finally, the transfer time  $\delta t$  is defined as

$$\delta t = t_2 - t_1 \quad (97)$$

and its derivative is defined as

$$d\delta t = dt_2 - dt_1 \quad (98)$$

Therefore, using equations (98) and (96) in equation (95), the differential cost function is obtained for a time-open, two-impulse transfer between fixed initial and final states.

$$dJ = -H d\delta t \quad (99)$$

The classical condition for a time-open trajectory between fixed states is that the Hamiltonian function be equal to zero. Equation (99) indicates that the cost function  $J$  will be minimized when  $H \rightarrow 0$ .

### Orbit-to-Orbit Transfer

Consider a transfer between two orbits, both fixed in shape and in orientation, for which the transfer time and the departure and arrival true anomalies  $\theta_1$  and  $\theta_2$ , respectively, are open (ref. 9) (fig. 10). What is the differential cost function for this solution? The differentials of the initial and final position and velocity vectors can be expressed as

$$\left. \begin{aligned} dR_1 &= V_1^- d\tau_1 & dR_2 &= V_2^+ d\tau_2 \\ dV_1 &= \dot{V}_1^- d\tau_1 & dV_2 &= \dot{V}_2^+ d\tau_2 \end{aligned} \right\} \quad (100)$$

wherein  $\tau$  is a time measurement in the initial or final orbit. The differential of this time  $d\tau$  is related to the differential in true anomaly  $d\theta$  by the conservation of



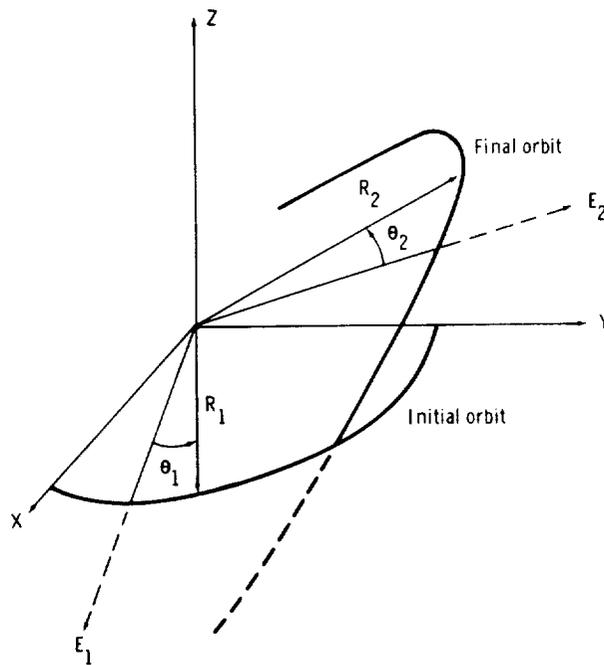


Figure 10.- Transfer between two orbits.

angular momentum (A.M.). The magnitude of the A.M. vector in the two orbits may be written as

$$\left. \begin{aligned} h_1 &= \left( R_1^T R_1 \right) \frac{d\theta_1}{d\tau_1} \\ h_2 &= \left( R_2^T R_2 \right) \frac{d\theta_2}{d\tau_2} \end{aligned} \right\} \quad (101)$$

By solving for  $d\tau_1$  and  $d\tau_2$  from equation (101) and using the results in equation (100), one obtains the position and velocity differentials

$$\left. \begin{aligned} dR_1 &= V_1 - \frac{\left( R_1^T R_1 \right)}{h_1} d\theta_1 & dR_2 &= V_2 + \frac{\left( R_2^T R_2 \right)}{h_2} d\theta_2 \\ dV_1 &= -\frac{\mu}{|R_1| h_1} R_1 d\theta_1 & dV_2 &= -\frac{\mu R_2}{|R_2| h_2} d\theta_2 \end{aligned} \right\} \quad (102)$$

wherein  $\mu$  is the gravitational parameter and use has been made of the acceleration vector definition. Using equation (102) in equation (90) results in

$$\begin{aligned}
 dJ = & H_1 dt_1 + \frac{1}{h_1} \left[ \frac{\mu}{|R_1|} \lambda_1^T R_1 + \left( R_1^T R_1 \right) \dot{\lambda}_1^T V_1^- \right] d\theta_1 \\
 & - H_2 dt_2 - \frac{1}{h_2} \left[ \frac{\mu}{|R_2|} \lambda_2^T R_2 + \left( R_2^T R_2 \right) \dot{\lambda}_2^T V_2^+ \right] d\theta_2 \\
 & + \left( \dot{\lambda}_m^+ - \dot{\lambda}_m^- \right)^T dR_m + \left( H_m^+ - H_m^- \right) dt_m
 \end{aligned} \tag{103}$$

The coefficients of  $d\theta_1$  and  $d\theta_2$  in equation (103) are  $\partial J / \partial \theta_1$  and  $\partial J / \partial \theta_2$ , respectively.

### Generalized Rendezvous

As a final example, determine the differential cost function for the generalized rendezvous (G.R.) problem. A G.R. problem is characterized by an open-time transfer that is functionally dependent on the motion in the initial or final orbit. If the initial and final position and velocity vectors are given at the same time  $t_1$ , then

1.  $[R_1(t_1), V_1(t_1), t_1]$
2.  $[R_2(t_1), V_2(t_1), t_1]$

If they are also initiated with a transfer time  $\delta t = t_2 - t_1$ , then the final position and velocity vectors are described as  $[R_2(t_2), V_2(t_2), t_2]$ . If the transfer time is incremented by  $\Delta \delta t$  to  $\delta t'$ , then  $\delta t' = \delta t + \Delta \delta t$  and the new final position and velocity vectors are described as  $[R_2(t_2 + \Delta \delta t), V_2(t_2 + \Delta \delta t), t_2 + \Delta \delta t]$ . Rewriting the general differential cost function (eq. (90)) results in the following equation wherein the superscript ' refers to differentiation with respect to  $\tau$ .

$$\begin{aligned}
 dJ = & \left( \lambda_1^T \dot{V}_1^- - \dot{\lambda}_1^T V_1^+ \right) dt_1 - \left( \lambda_1^T V_1' - \dot{\lambda}_1^T R_1' \right) d\tau_1 \\
 & - \left( \lambda_2^T \dot{V}_2^- - \dot{\lambda}_2^T V_2^- \right) dt_2 + \left( \lambda_2^T V_2' - \dot{\lambda}_2^T R_2' \right) d\tau_2 \\
 & + \left( \dot{\lambda}_m^+ - \dot{\lambda}_m^- \right)^T dR_m + \left( H_m^+ - H_m^- \right) dt_m
 \end{aligned} \tag{104}$$

In a G.R. problem, however, changes in time in the initial or final orbit  $\tau$  are equal to changes in the initial or final time, or

$$\left. \begin{aligned} d\tau_1 &= dt_1 \\ d\tau_2 &= dt_2 \end{aligned} \right\} \quad (105)$$

Thus,  $V_1' d\tau_1 = \dot{V}_1 dt_1$  et cetera. By rewriting equation (104), making use of equation (105), the following expression is obtained.

$$\begin{aligned} dJ &= -\dot{\lambda}_1^T (V_1^+ - V_1^-) dt_1 + \dot{\lambda}_2^T (V_2^- - V_2^+) dt_2 \\ &+ \left( \dot{\lambda}_m^+ - \dot{\lambda}_m^- \right)^T dR_m + \left( H_m^+ - H_m^- \right) dt_m \end{aligned} \quad (106)$$

But

$$\left. \begin{aligned} (V_1^+ - V_1^-) &= \Delta V_1 \\ (V_2^+ - V_2^-) &= \Delta V_2 \end{aligned} \right\} \quad (107)$$

and the impulses are in the direction of the multipliers

$$\left. \begin{aligned} \Delta V_1 &= |\Delta V_1| \lambda_1 \\ \Delta V_2 &= |\Delta V_2| \lambda_2 \end{aligned} \right\} \quad (108)$$

Using equations (107) and (108) in equation (106) gives

$$\begin{aligned} dJ &= -|\Delta V_1| \left( \lambda_1^T \dot{\lambda}_1 \right) dt_1 - |\Delta V_2| \left( \lambda_2^T \dot{\lambda}_2 \right) dt_2 \\ &+ \left( \dot{\lambda}_m^+ - \dot{\lambda}_m^- \right)^T dR_m + \left( H_m^+ - H_m^- \right) dt_m \end{aligned} \quad (109)$$

as the differential cost function for the generalized rendezvous problem. Note that for optimal departure and arrival times from the initial and final orbits, the primer and its derivative must be orthogonal, or  $\lambda \perp \dot{\lambda}$  (G.R.). This condition, as previously discussed, is exactly the condition on the primer vector derivative at an intermediate impulse point (eq. (32) for the nonimpulsive case). A primer history for an optimal rendezvous with, for instance, three impulses may appear as shown in figure 11.

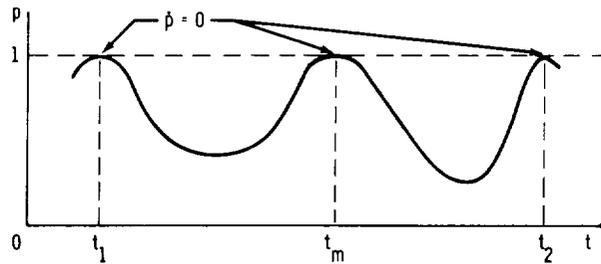


Figure 11.- Optimal three-impulse rendezvous.

That the primer and its derivative must also be orthogonal at an intermediate impulse point ( $t = t_m$ ) can be proved easily. On an optimal solution, the Hamiltonian function  $H$  must be continuous, or

$$\left. \begin{aligned} H_m^- &= H_m^+ \\ \lambda_m^T \dot{V}_m^- - \dot{\lambda}_m^T V_m^- &= \lambda_m^T \dot{V}_m^+ - \dot{\lambda}_m^T V_m^+ \end{aligned} \right\} \quad (110)$$

wherein  $\lambda_m$  has been taken to be continuous. Now

$$\dot{V}_m^- = \dot{V}_m^+ \quad (111)$$

Therefore, equation (110) resolves to

$$-\dot{\lambda}_m^T V_m^- = \dot{\lambda}_m^T V_m^+ \quad (112)$$

But, on an optimal solution,  $\dot{\lambda}$  is continuous, or

$$\dot{\lambda}_m^T (V_m^+ - V_m^-) = 0 \quad (113)$$

But

$$V_m^+ - V_m^- = |\Delta V_m| \lambda_m \quad (114)$$

Thus,

$$|\Delta V_m| \dot{\lambda}_m^T \lambda_m = 0 \quad (115)$$

or  $\lambda_m \perp \dot{\lambda}_m$ .

## INEQUALITY CONSTRAINTS IN PRIMER OPTIMAL N-IMPULSE SOLUTIONS

The material in this section is based on reference 4.

### Periapsis Constraint

Consider the following sample problem.

1. Depart from the initial position and velocity vectors  $R_1$  and  $V_1$  at the time  $t_1$ .
2. Arrive at the final position and velocity vectors  $R_2$  and  $V_2$  at the time  $t_2$ .
3. Problem: Find the optimum number of impulses and the minimum cost (eq. (47)) while requiring that the radius of periapsis of any transfer conic  $r_{p_i}$  of the N-impulse solution be larger than some value  $r_c$ , or  $r_{p_i} - r_c \geq 0$  (fig. 12).

To use the penalty function approach, consider a function

$$J_P = p e^{1-\mu^n} \quad (116)$$

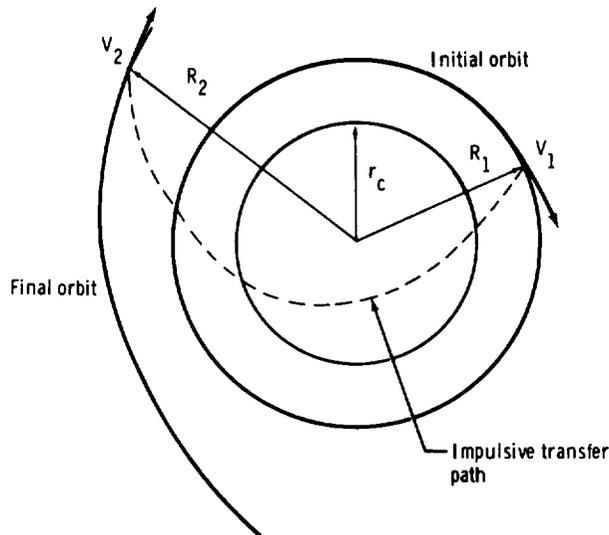


Figure 12.- Radius-of-periapsis inequality constraint.

wherein, as in figure 13,  $p$  is a positive constant,  $n$  is a suitably large positive integer, and  $\mu$  is

$$\mu = \frac{r}{r_c} p \quad (117)$$

If the penalty cost for violating the constraint is designated as  $J_p$  and  $dJ_p/d\mu$  as  $J_p'$ , then the following relationships apply.

1. When  $\mu \gg 1$ ,  $J_p \approx 0$  and  $J_p' \approx 0$ .
2. When  $\mu = 1 + \epsilon$ ,  $J_p > 0$  and  $J_p' \ll 0$ .
3. When  $0 < \mu \leq 1$ ,  $J_p > 0$  and  $J_p' < 0$ .

The symbol  $\epsilon$  represents a small positive number. Note that  $J_p$  and  $J_p'$  remain finite for  $\mu = 1$  because  $J_p$  and  $J_p'$  will be required to be continuous in the interval  $0 < \mu \leq 1$ . Note that in this interval,  $J_p' < 0$ , or that the parameter  $\mu$  will move in a direction that satisfies the constraint. If  $n$  is chosen sufficiently large, the parameter  $\mu$  may be allowed to approach the constrained value as closely as desired. The parameter  $J_p$  is a single-walled function because  $\mu$  is constrained to take on

values only to the right of unity. A double-walled function (cost well) may be described by first creating a mirror function  $J_q$

$$J_q = qe^{1-\delta^m} \tag{118}$$

wherein

$$\delta = a/\mu \tag{119}$$

when  $a > 1$ . If  $p = q$  and  $n = m$ , then the maximum width of the well is  $a - 1$  as shown in figure 14. The cost well function is defined as

$$J_W = J_p + J_q \tag{120}$$

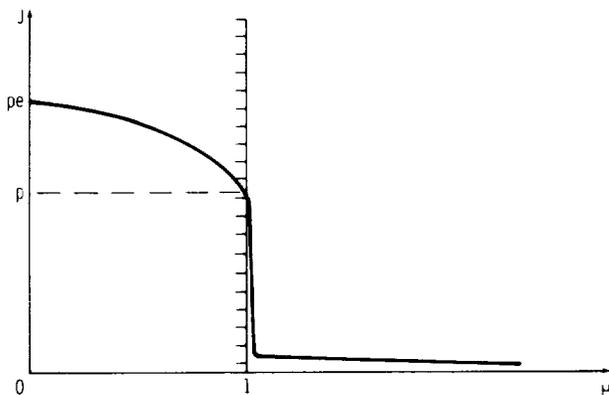


Figure 13.- Single-wall penalty function.

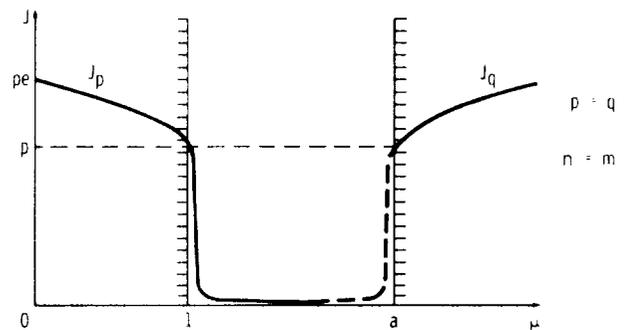


Figure 14.- Double-wall penalty function (cost well).

For a minimum penalty cost,  $1 < \mu < a$ .

The cost on an optimal N-impulse solution when inequality constraints are imposed on the transfer conics is

$$J = J_0 + J_W \tag{121}$$

wherein

$$J_0 = \sum_{i=1}^N |\Delta V_i| \quad (122)$$

and

$$J_W = \sum_{i=1}^{N-1} \left[ p_i e^{1-\mu_i^n} + q_i e^{1-\delta_i^m} \right] \quad (123)$$

For simplicity of nomenclature, consider a three-impulse solution ( $N = 3$ ). Then, the optimal differential cost can be written as

$$dJ_0 = \frac{\partial J_0}{\partial R_m} dR_m + \frac{\partial J_0}{\partial t_m} dt_m + \frac{\partial J_0}{\partial S_1} dS_1 + \frac{\partial J_0}{\partial S_2} dS_2 \quad (124)$$

wherein  $dS_1$  and  $dS_2$  represent possible differential changes in the initial and final states  $(R, V, t)$ .

Because the penalty function  $J_W$  is only a function of  $\mu$ , the differential is

$$dJ_W = \frac{\partial J_W}{\partial \mu_1} d\mu_1 + \frac{\partial J_W}{\partial \mu_2} d\mu_2 \quad (125)$$

wherein the subscripts 1 and 2 refer to the first and second conics, respectively. But  $\mu$  is a function of  $R$  and  $V$  only on the conics; because the constraint can be evaluated anywhere on these conics, it is convenient to choose the intermediate impulse time  $t_m$ .

Consider only the penalty cost on the first conic (dropping the subscripts 1 and 2).

$$dJ_W = \frac{\partial J_W}{\partial \mu} \left( \frac{\partial \mu}{\partial R_m} dR_m + \frac{\partial \mu}{\partial V_m} dV_m \right) \quad (126)$$



The term  $dR_m$  requires no further change. The term  $dV_m^-$  must be expressed in terms of the control parameters of the optimal N-impulse solution ( $dR_m, dt_m, dS_1, dS_2$ ). Expanding  $dV_m^-$  results in

$$dV_m^- = \frac{\partial V_m^-}{\partial R_m} dR_m + \frac{\partial V_m^-}{\partial t_m} dt_m \quad (127)$$

which may be written as

$$dV_m^- = \delta V_m^- + \dot{V}_m^- dt_m \quad (128)$$

From the state transition matrix, the following expression may be obtained.

$$\begin{bmatrix} \delta R(t_1) \\ \delta V(t_1) \end{bmatrix} = \Phi(t_1, t_m^-) \begin{bmatrix} \delta R_m^- \\ \delta V_m^- \end{bmatrix} \quad (129)$$

Solving for  $\delta V_m^-$  results in

$$\delta V_m^- = \varphi_{12}^{-1} (\delta R_1^+ - \varphi_{11} \delta R_m^-) \quad (130)$$

Now, the perturbations in the position vectors at  $t_1^+$  and  $t_m^-$  are

$$\left. \begin{aligned} \delta R_1^+ &= dR_1 - V_1^+ dt_1 \\ \delta R_m^- &= dR_m^- - V_m^- dt_m \end{aligned} \right\} \quad (131)$$

Using equation (131) in equation (130) results in

$$\delta V_m^- = \varphi_{12}^{-1} \left[ dR_1 - V_1^+ dt_1 - \varphi_{11} (dR_m^- - V_m^- dt_m) \right] \quad (132)$$

Using equation (132) in equation (128) results in the following equation for the differential  $dV_m^-$ .

$$dV_m^- = \varphi_{12}^{-1} \left[ dR_1 - V_1^+ dt_1 - \varphi_{11} (dR_m - V_m^- dt_m) \right] + \dot{V}_m^- dt_m \quad (133)$$

The differential change in the penalty function (eq. (126)) for the first conic of the three-impulse solution is

$$dJ_W = \frac{\delta J_W}{\partial \mu} \left\{ \left( \frac{\partial \mu}{\partial R_m} - \frac{\partial \mu}{\partial V_m^-} \varphi_{12}^{-1} \varphi_{11} \right) dR_m + \frac{\partial \mu}{\partial V_m^-} \left[ \left( \dot{V}_m^- + \varphi_{12}^{-1} \varphi_{11} V_m^- \right) dt_m + \varphi_{12}^{-1} (dR_1 - V_1^+ dt_1) \right] \right\} \quad (134)$$

The total differential cost on an optimal three-impulse solution subject to an inequality constraint on the first conic is the sum of equations (124) and (134). For various boundary conditions, the equations are modified as follows.

1. Initial departure time and position vector fixed

$$\left. \begin{aligned} dt_1 &= 0 \\ dR_1 &= 0 \end{aligned} \right\} \quad (135)$$

2. Initial departure time fixed; initial true anomaly free

$$\left. \begin{aligned} dt_1 &= 0 \\ dR_1 &= \frac{\begin{pmatrix} R_1 & T \\ R_1 & R_1 \end{pmatrix}}{h_1} V_1^- d\theta_1 \end{aligned} \right\} \quad (136)$$

3. Generalized rendezvous

$$dR_1 = V_1^- dt_1 \quad (137)$$

The following is an example involving a radius-of-periapsis constraint. The magnitude of the radius of periapsis is

$$r_p = h^2 / (1 + e) \quad (138)$$

wherein  $h$  and  $e$  are the magnitudes of the angular momentum  $H$  and the periapsis vector  $E$ , respectively. The parameter  $\mu$  is

$$\mu = \frac{1}{r_c} \left( \frac{h^2}{1 + e} \right) \quad (139)$$

Rewriting equation (139) results in

$$\mu = \frac{1}{r_c} \frac{H^T H}{1 + |E|} \quad (140)$$

wherein

$$H = R \times V \quad (141)$$

$$E = V \times H - \mu_m R / r \quad (142)$$

and  $\mu_m$  is the gravitational parameter. Because  $H$  and  $E$  can be evaluated at  $t_m^-$ , the partial derivatives required in equation (134) can be expressed as

$$\left. \begin{aligned} \frac{\partial \mu}{\partial R_m} &= \frac{\partial \mu}{\partial H} \frac{\partial H}{\partial R_m} + \frac{\partial \mu}{\partial E} \frac{\partial E}{\partial R_m} \\ \frac{\partial \mu}{\partial V_m} &= \frac{\partial \mu}{\partial H} \frac{\partial H}{\partial V_m} + \frac{\partial \mu}{\partial E} \frac{\partial E}{\partial V_m} \end{aligned} \right\} \quad (143)$$

## Lighting Constraint

As a final example, consider the problem of requiring an impulse to occur within a given lighting condition (fig. 15). The inertial frame (X,Y,Z) is located with respect to the Sun by the vector S. If the position vector R at the time of constraint applicability is oriented with respect to the Sun vector by the angle  $\psi$ , then the function  $\mu$  for a lighting inequality constraint can be defined as  $\mu = \psi/\psi_{\min}$ , wherein  $\psi_{\min}$  and  $\psi_{\max}$  will describe the minimum and maximum bounds of the constraint. From figure 15, the angle  $\psi$  may be expressed as

$$\psi = \cos^{-1}(\ell_R^T \ell_S) \quad (144)$$

wherein  $\ell_R$  and  $\ell_S$  are unit vectors in the R and S vector directions, respectively.

The optimal impulsive solution requires not only the cost function J but also its gradient  $\nabla J$ . The total derivative of the cost function J is

$$dJ = \frac{\partial J}{\partial \mu} \frac{\partial \mu}{\partial R} dR \quad (145)$$

From the definitions given, the partial of  $\mu$  with respect to the vector R is

$$\frac{\partial \mu}{\partial R} = \frac{\ell_S^T (\ell_R \ell_R^T - I)}{|\mathbf{R}| \psi_{\min} \sin \psi} \quad (146)$$

wherein I is the three-by-three identity matrix. From the conservation of angular momentum, the differential of R may be expressed as

$$dR = \frac{(\mathbf{R}^T \mathbf{R})}{|H|} \mathbf{V} d\theta \quad (147)$$

wherein  $\theta$  is the true anomaly and H is the angular momentum vector. Thus, the cost function J and its gradient can be readily expressed. The only question remaining is how to determine the quadrant for the angle  $\psi$ .

If  $\psi$  is measured positive by a counterclockwise rotation of the vector R into the vector S (fig. 15), then the following test will define the quadrant for the angle  $\psi$ .



1. The expression  $\psi = \psi, (\mathbf{R} \times \mathbf{V})^T (\mathbf{R} \times \mathbf{S}) > 0$  implies that the Sun is leading the vector  $\mathbf{R}$ .

2. The expression  $\psi = 2\pi - \psi, (\mathbf{R} \times \mathbf{V})^T (\mathbf{R} \times \mathbf{S}) < 0$  implies that the Sun is lagging the vector  $\mathbf{R}$ .

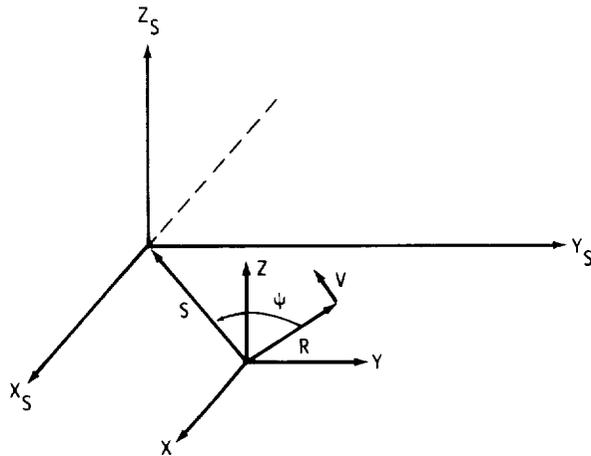


Figure 15.- Coordinate system for lighting inequality constraint.

### CONCLUDING REMARKS

The necessary conditions that must be satisfied by the primer vector and its derivative on a general, two-body, N-impulse optimal trajectory have been developed. Example problems have been presented to illustrate a variety of equality and inequality constraints on the solution. The Optimal, Multi, Impulse Rendezvous computer program was produced to encompass this general solution.

Lyndon B. Johnson Space Center  
National Aeronautics and Space Administration  
Houston, Texas, May 15, 1975  
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